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Galilean covariance and non-relativistic Bhabha equations

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Abstract

We apply a five-dimensional formulation of Galilean covariance to construct non-relativistic Bhabha first-order wave equations which, depending on the representation, correspond either to the well known Dirac equation (for particles with spin $1/2$) or the Duffin–Kemmer–Petiau equation (for spinless and spin 1 particles). Here the irreducible representations belong to the Lie algebra of the 'de Sitter group' in $4+1$ dimensions, $SO(5,1)$. Using this approach, the non-relativistic limits of the corresponding equations are obtained directly, without taking any low-velocity approximation. As a simple illustration, we discuss the harmonic oscillator.

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1. Introduction

The purpose of this paper is to review and complete an earlier work, where a geometrical formulation of Galilean covariance in five dimensions was used to construct the Duffin–Kemmer–Petiau (DKP) wave equation for spinless particles [1]. Here we emphasize the role played by the underlying Lie algebra $so(5,1)$, and construct the corresponding Bhabha equations describing non-relativistic spin 0, $1/2$ and 1 particles.

During the last decades, many concepts and techniques of quantum field theory have been shared between relativistic particle physics and condensed matter physics: gauge invariance, spontaneous symmetry breaking, Goldstone bosons, and so on. However, whereas the first is based on Lorentz-covariant relativity, a key ingredient to describe some topics of condensed matter is rather the Galilei covariance. This has prompted a geometrical formulation of the Galilei covariance so that the methods would follow the same general lines as the usual relativistic theories, in particular the possibility of using the powerful techniques already existing in the relativistic situations, e.g. in quantum field theory. Therefore, a more geometrical description of the Galilean invariance, based on a metric space, than what is usually found in the literature, is needed. (For an excellent, albeit not recent, review of Galilean invariance see [2].) Once a covariant formalism is at hand, then physical applications involve the construction of invariant wave equations analogous to the Dirac or Maxwell equations in Lorentz-invariant quantum field theory, obtaining thereby their non-relativistic version directly. This paper follows this direction. In the rest of this section we briefly review the notion of Galilean covariance in five dimensions [3–7]. In addition the relativistic Bhabha field equation, along with some of its symmetry properties, is reviewed.

A way to achieve a covariant formulation of Galilean invariance is to extend the ordinary space–time by adding an extra dimension so that the formalism involves an embedding in a *five*-dimensional de Sitter space of type 4 + 1. One obtains thereby a covariant form for non-relativistic theories. The precise formulation of the Galilei transformations involving the extra dimension has been introduced by Takahashi [3] and investigated in relation with the Schrödinger field [4]. A simple dynamical argument in favour of working in five dimensions is the following. The Galilei transformations

$$\begin{aligned} \boldsymbol{x} &\to \boldsymbol{x}' = R\boldsymbol{x} - \boldsymbol{v}t + \boldsymbol{a} \\ t &\to t' = t + b \end{aligned} \tag{1}$$

clearly do not leave the free Lagrangian

$$L = \frac{1}{2}m\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)^2 \tag{2}$$

invariant, although they do so for the equations of motion:

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{0}. \tag{3}$$

A well known consequence of this quasi-invariance is that the corresponding quantum wavefunction is given by a projective representation of the Galilei group. This follows from the fact that the transformed Lagrangian differs from the original Lagrangian by a total derivative, and since the wavefunction is related to $\exp\frac{\mathrm{i}}{\hbar}\int L\,\mathrm{d}t$, the additional term induces a phase factor in the wavefunction. However, following an earlier idea [6], full invariance is achieved if one enlarges the configuration space (here by one dimension) and defines

$$L \to L - m\frac{\mathrm{d}s}{\mathrm{d}t} \tag{4}$$

given that the extra parameter s transforms as

$$s \to s - (R\boldsymbol{x})\cdot\boldsymbol{v} + \tfrac{1}{2}\boldsymbol{v}^2 t + \text{const}. \tag{5}$$

Therefore one obtains a Galilei-covariant formalism as shown in [3, 4]. The transformations given in equations (1) and (5) leave invariant the scalar product $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, with the metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{6}$$

for a five-dimensional manifold with coordinates

$$x = (x^1, x^2, x^3, x^4, x^5) = (\boldsymbol{x}, t, s). \tag{7}$$

In the following we shall refer to equation (6) as the *Galilean metric*. As mentioned in [4] this metric can be diagonalized into diag(+, +, +, +, −). Thus, the Galilean covariant formalism consists in embedding the usual three-dimensional space (with time t playing the role of an external parameter) into a 4 + 1 de Sitter space. We shall see below that this embedding has its counterpart at the level of fields and, in our opinion, this is where the most fundamental mathematical problems are left unexplored: one needs a classification of such embeddings and well defined equivalence relations among them. Here we use the above formalism to construct and investigate the *non-relativistic* Bhabha equations that describe particles of spin 0, 1/2 and 1, that is, first-order linear wave equations as investigated earlier for relativistic fields [8–10].

Dirac, Fierz and Pauli constructed wave equations using the guiding principle that each component satisfies a dispersion relation, e.g. the second-order Klein–Gordon equation [11, 12]. Such requirements (a first-order linear equation and the dispersion relation) lead to commutation relations among the generators appearing in the wave equation, e.g. the Clifford algebra of the Dirac gamma matrices. For spin greater than 1, these equations cannot be written in the form of equation (8), as shown in [8]. Then the equations obtained by Dirac, Fierz and Pauli involve an additional set of subsidiary conditions which clearly become more complex when interactions are introduced. (Let us just mention an alternative procedure, due to Majorana [13], in which the existence of the dispersion relation is not required.)

The various first-order wave equations in 3 + 1 space–time may be considered as particular cases of the generic Bhabha equations [8–10]

$$(\alpha^{\mu}\partial_{\mu} + k)\Psi = 0 \tag{8}$$

where the α^{μ} are given by the generators of the irreducible representations of $so(4, 1)$

$$\alpha_{\mu} \equiv J_{\mu 5} \qquad J_{\mu\nu} = -\mathrm{i}\left[\alpha_{\mu}, \alpha_{\nu}\right] \qquad J_{55} = 0 \tag{9}$$

and k is a constant. Depending on the specific irreducible representation of $so(4, 1)$, equation (8) is, for instance, the Dirac [11] or the DKP [14] equation, corresponding to half integer and integer spin, respectively. The DKP ring is discussed in more details in [9, 15].

Unlike the Dirac equation, the DKP equation has had a mixed success; it has been used to describe the interaction of charged particles with an electromagnetic field and the results agreed (up to one-loop corrections) with those based on a second-order formalism. However it is not clear whether the equivalence between the two formalisms is valid to all orders. Among the recent applications of DKP equations, let us mention a *post mortem* article by Gribov [16], in which he investigated QCD by using the DKP formalism to introduce multi-component Green functions for the gluon, thereby presenting a form in which the interaction is momentum independent. More recently, a proof of the equivalence between DKP and Klein–Gordon theories has been provided by Fainberg and Pimentel [17] for charged scalar particles interacting with external and quantized electromagnetic and Yang–Mills fields, as well as with an external gravitational field. Along the same lines, Lunardi *et al* [18] have solved two problems associated with the gauge invariance of the DKP field: the occurrence of an anomalous term in the Hamiltonian form and a difference between the interaction terms in DKP and Klein–Gordon Lagrangians. Let us mention also the recent work by Kanatchikov [19], related to field theories admitting a De Donder–Weyl Hamiltonian formulation.

Nikitin and Fuschich [20] have attempted to incorporate the ideas of Bhabha to equations for higher spins in non-relativistic physics. The present paper has the goal to accomplish this objective within a fully Galilean covariant approach. This is achieved by using equation (8) as a starting point with the metric given by equation (6). Hereafter, we follow the general Bhabha theory in the five-dimensional space and use representations of $so(5, 1)$ to construct the α of equation (8). Since the representations of the Lie algebra $so(5, 1)$ are so crucial, let us give some details about their theory. The compact real form of $so(5, 1)$ is $so(6)$. As is well known [21], the Lie algebra $so(6)$, and its non-compact real forms, are 15-dimensional with rank three. Using Cartan notation, the complexification of $so(6)$ is $D_3 \approx A_3$ which implies that $so(6)$ is isomorphic to $su(4)$. Therefore its unitary irreducible representations include the four-dimensional fundamental representation of $su(4)$, as well as the six-dimensional fundamental representation of $so(6)$. Both are used hereafter. Using the Gelfand and Zeitlin notation [9,22], the irreducible representations of $so(5,1)$ are denoted by triplets $\{l_1, l_2, l_3\}$ of integers or half-integers such that $l_1 \geqslant l_2 \geqslant l_3$. Their dimensions are given by

$$\dim\{l_1, l_2, l_3\} = \tfrac{1}{12}[(l_1+2)^2 - (l_2+1)^2][(l_1+2)^2 - l_3^2][(l_2+1)^2 - l_3^2] \tag{10}$$

and they describe particles with spin at most equal to l_1. For instance the dimensions of the irreducible representations $\{0,0,0\}$, $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, $\{1,0,0\}$, $\{1,1,1\}$, $\{1,1,0\}$, $\{2,0,0\}$ and $\{2,2,2\}$ are 1, 4, 6, 10, 15, 20 and 35, respectively. Hereafter we shall use 4-, 6- and 15-dimensional irreducible representations to describe fields with spin 1/2, 0 and 1, respectively. In [20], the 10-dimensional representation is used for spin 1. As discussed in [8, 10] for the relativistic cases, the Bhabha formalism typically involves multimass and multispin systems, and a detailed description of the spin content of each representation (of $so(6)$) remains to be done for the non-relativistic theories.

2. Non-relativistic Bhabha equations and $so(5,1)$

In relativistic field theories, the Bhabha equation is invariant under the inhomogeneous Lorentz transformations. To the generators of these transformations we can add the four generators α of equation (8) to extend the Lie algebra $so(3,1)$ to the algebra $so(4,1)$, which is then a unifying Lie algebra in the sense that its various representations provide the generators of the linear wave equations for different spins [8]. Similarly, the Galilean covariance of the non-relativistic Bhabha equations, equation (8), hereafter is extended to the Lie algebra $so(5,1)$ by including the five generators α. Therefore the non-relativistic Bhabha wave equations are built using the irreducible representations of $so(5,1)$.

The calculations described hereafter, within the scheme of Galilean covariance, are similar to the relativistic case [8, 10]. We require the Bhabha equation, (8), where μ now runs from 1 to 5, to be invariant under Galilei boosts, given by equations (1) and (5) with $R^i_j = \delta^i_j$, $\boldsymbol{a} = \boldsymbol{0}$ and $b = 0$, cast together into the form

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{11}$$

with x given in equation (7). This implies that the field Ψ transforms according to

$$\Psi'(x) = U(\Lambda)\Psi(\Lambda^{-1}x) \tag{12}$$

and the generators α transform as

$$[U(\Lambda)]^{-1}\alpha^\mu U(\Lambda) = \Lambda^\mu_\nu \alpha^\nu \tag{13}$$

just like the usual relativistic wave equations.

Writing the generators of the transformations given in equation (11) as

$$J^{\mu\nu} \equiv \mathrm{i}\,(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{14}$$

they are found to satisfy the usual commutations relations

$$[J^{\mu\nu}, J^{\alpha\beta}] = \mathrm{i}\left(g^{\nu\alpha} J^{\mu\beta} + g^{\mu\beta} J^{\nu\alpha} - g^{\nu\beta} J^{\mu\alpha} - g^{\mu\alpha} J^{\nu\beta}\right) \tag{15}$$

where $g^{\mu\nu}$ is the Galilean metric, equation (6). With $U(\Lambda) = \exp(\mathrm{i}\omega_{\mu\nu} J^{\mu\nu})$, where ω is given by $\Lambda_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$, one obtains from equation (13) that

$$[\alpha^\mu, J^{\alpha\beta}] = \mathrm{i}(g^{\mu\alpha}\alpha^\beta - g^{\mu\beta}\alpha^\alpha). \tag{16}$$

The next step, following the same procedure as the first paper of [8], consists in obtaining

$$\left[[\alpha^\mu, \alpha^\nu], J^{\alpha\beta}\right] = \mathrm{i}\left(g^{\nu\alpha}[\alpha^\mu, \alpha^\beta] + g^{\mu\beta}[\alpha^\nu, \alpha^\alpha] - g^{\nu\beta}[\alpha^\mu, \alpha^\alpha] - g^{\mu\alpha}[\alpha^\nu, \alpha^\beta]\right) \tag{17}$$

which upon comparison with equation (15) shows that $J^{\mu\nu}$ is proportional to the commutator $[\alpha^\mu, \alpha^\nu]$, and we shall choose the constant of proportionality to be $-\mathrm{i}$, as in equation (9), so that

$$J^{\mu\nu} = -\mathrm{i}\left[\alpha^\mu, \alpha^\nu\right]. \tag{18}$$

Equations (15)–(17) together form the Lie algebra $so(5, 1)$ and it is convenient to rewrite them using the uniform notation

$$\{J^{AB} \mid A, B = 1, \ldots, 6;\ A < B\} \tag{19}$$

with

$$J^{AB} = J^{\mu\nu} \tag{20}$$

where both A and B lie between 1 and 5, and

$$J^{A6} = -J^{6A} = \alpha^A \qquad A = 1, \ldots, 5. \tag{21}$$

The additional ingredients that we need are the metric elements involving the index 6; we choose them to be

$$g^{66} = +1 \qquad \text{and} \qquad g^{\mu 6} = 0 \qquad \mu \neq 6. \tag{22}$$

Given this information, equations (15)–(17) can be all cast into the unified form

$$\left[J^{AB}, J^{CD}\right] = \mathrm{i}\left(g^{BC}J^{AD} + g^{AD}J^{BC} - g^{AC}J^{BD} - g^{BD}J^{AC}\right) \tag{23}$$

where the six-by-six metric is given by equation (6), supplemented with equation (22):

$$g_{AB} = g^{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

This metric clearly generates the group $SO(5, 1)$ since it can be diagonalized to $(+, +, +, +, -, +)$.

In the following, we shall use the momentum expression of equation (8):

$$\left(\alpha^\mu p_\mu - \mathrm{i}k\right)\Psi = \left(\boldsymbol{\alpha} \cdot \boldsymbol{p} + \alpha^4 p_4 + \alpha^5 p_5 - \mathrm{i}k\right)\Psi = 0. \tag{25}$$

The factor $\hbar$ has been absorbed into the constant k. The indices from 1 to 6 correspond to the metric, equation (24), and the generators of $so(5, 1)$. We use Greek indices to denote Galilei-covariant quantities, such as x^μ, α^μ, etc where $\mu = 1, \ldots, 5$. Lowercase letters $m, n, \ldots = 1, 2, 3$, denote ordinary space coordinates, as in x^n and J^{n6}.

3. DKP equation: spin zero and spin one

The DKP equation is [14]

$$(\beta^\mu \partial_\mu + k)\Psi = 0 \tag{26}$$

where k is an arbitrary constant that we do not specify at this point. In the non-relativistic regime considered here, the five matrices β satisfy the DKP algebra

$$\beta^\mu \beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda \beta^\mu = g^{\mu\lambda}\beta^\nu + g^{\nu\lambda}\beta^\mu \tag{27}$$

where $g^{\mu\nu}$ is the Galilean metric. (In the relativistic case, there are only four generators β, and $g^{\mu\nu}$ is the usual four-dimensional Lorentz metric.)

The DKP equation (26) is obtained from the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{DKP}} &= \overline{\Psi}\left[\tfrac{1}{2}\beta^\mu(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) + k\right]\Psi \\ &= \tfrac{1}{2}\overline{\Psi}\beta^\mu\partial_\mu\Psi - \tfrac{1}{2}\left(\partial_\mu\overline{\Psi}\right)\beta^\mu\Psi + k\overline{\Psi}\Psi.\end{aligned} \tag{28}$$

The adjoint of Ψ is given by $\overline{\Psi} \equiv \Psi^\dagger\eta$, where

$$\eta = (\beta^4 + \beta^5)^2 + \mathbf{1}. \tag{29}$$

One can show that

$$\begin{aligned}\beta^m\eta &= -\eta\beta^m \\ \beta^4\eta &= \eta\beta^5 \\ \beta^5\eta &= \eta\beta^4\end{aligned} \tag{30}$$

for $m = 1, 2, 3$.

The Lagrangian given by equation (28) admits a conserved five-current

$$j^\mu_{\mathrm{DKP}} = \overline{\Psi}\beta^\mu\Psi. \tag{31}$$

By developing the sum

$$0 = \partial_\mu j^\mu_{\mathrm{DKP}} = \boldsymbol{\nabla}\cdot\left(\overline{\Psi}\boldsymbol{\beta}\Psi\right) + \partial_4\left(\overline{\Psi}\beta^4\Psi\right) + \partial_5\left(\overline{\Psi}\beta^5\Psi\right) \tag{32}$$

one can identify: $\boldsymbol{j}_{\mathrm{DKP}} \equiv (\overline{\Psi}\boldsymbol{\beta}\Psi)$, $\rho_{\mathrm{DKP}} \equiv (\overline{\Psi}\beta^4\Psi)$ and $j^5_{\mathrm{DKP}} \equiv (\overline{\Psi}\beta^5\Psi) = 0$.

3.1. DKP equation for spin zero

To construct the DKP equation for spinless particles we need five matrices of dimension six. Let us choose them as follows [1]:

$$\beta^1 = \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0 \end{pmatrix} \qquad \beta^2 = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0 \end{pmatrix} \qquad \beta^4 = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&-1&0 \end{pmatrix} \tag{33}$$

$$\beta^5 = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&-1&0&0 \end{pmatrix}.$$

They can be shown to generate the Lie algebra $so(5,1)$, by using equation (21):

$$J^{\mu 6} \equiv \beta^\mu \qquad \mu = 1, \ldots, 5. \tag{34}$$

From these elements, one can generate the remaining elements of $so(5, 1)$, and verify that they all satisfy equation (23). It is straightforward to verify that the matrices in equation (33) satisfy the DKP algebra given by equation (27) with $g^{\mu\nu}$ the Galilean metric in equation (6).

Hereafter we review the results first obtained in [1] for the sake of completeness, using the uniform procedure and notation described previously. The results therein agree with the non-relativistic limit of the equations obtained in [23]. We use the momentum expression of equation (26) given by equation (25) with the α replaced by the β of equation (33). If we introduce a DKP spinor

$$\Psi \equiv \begin{pmatrix} \boldsymbol{A} \\ \theta \\ \varphi \\ \phi \end{pmatrix} \tag{35}$$

with $\boldsymbol{A} = (A_x, A_y, A_z)$, then equation (25) leads to

$$\begin{aligned} &-\mathrm{i}k\boldsymbol{A} + \boldsymbol{p}\phi = \boldsymbol{0} \\ &-\mathrm{i}k\theta + p_4\phi = 0 \\ &-\mathrm{i}k\varphi + p_5\phi = 0 \\ &\boldsymbol{p}\cdot\boldsymbol{A} - p_5\theta - p_4\varphi - \mathrm{i}k\phi = 0. \end{aligned} \tag{36}$$

These equations can be expressed in terms of ϕ only as

$$\boldsymbol{p}^2\phi - 2p_4p_5\phi + k^2\phi = 0. \tag{37}$$

This becomes the Schrödinger equation

$$E\phi = \frac{\boldsymbol{p}^2}{2m}\phi \tag{38}$$

by defining the embedding

$$\boldsymbol{p} \to (\boldsymbol{p}, p_4, p_5) \qquad \text{such that} \quad p_4p_5 = mE \tag{39}$$

and absorbing the constant k into the energy as

$$E \to E - \frac{k^2}{2m}. \tag{40}$$

(The difference in sign with [1] is due to the factor i in equation (25).) Note that many definitions of p_4 and p_5 satisfying equation (39) are possible, because of the symmetry in p_4 and p_5 in the second term of equation (37). Such a symmetry may not exist in general, for instance with the spin half particles discussed in section 4.

Once a solution ϕ is known, then the DKP spinor, equation (35), can be written as

$$\Psi_{\text{free}} = \begin{pmatrix} -\mathrm{i}\boldsymbol{p}/k \\ -\mathrm{i}p_4/k \\ -\mathrm{i}mE/kp_4 \\ 1 \end{pmatrix}\phi \tag{41}$$

where the embedding of equation (39) has been used again.

Now let us turn to the harmonic oscillator. It is described by performing the non-minimal substitution

$$\boldsymbol{p} \to \boldsymbol{p} + \mathrm{i}\omega\eta\boldsymbol{r} \tag{42}$$

into equation (25), with η given by equations (29) and (33). (Later we will include a factor m into ω.) We thereby obtain

$$\begin{aligned} &-\mathrm{i}k\boldsymbol{A} + (\boldsymbol{p} - \mathrm{i}\omega\boldsymbol{r})\,\phi = \boldsymbol{0} \\ &\mathrm{i}k\theta = p_4\phi \\ &\mathrm{i}k\varphi = p_5\phi \\ &(\boldsymbol{p} + \mathrm{i}\omega\boldsymbol{r})\cdot\boldsymbol{A} - p_5\theta - p_4\varphi - \mathrm{i}k\phi = 0 \end{aligned} \tag{43}$$

which can be reduced to

$$(\boldsymbol{p} + \mathrm{i}\omega\boldsymbol{r})\cdot(\boldsymbol{p} - \mathrm{i}\omega\boldsymbol{r})\,\phi - p_5p_4\phi - p_4p_5\phi + k^2\phi = 0. \tag{44}$$

This equation can be developed as follows:

$$\boldsymbol{p}^2\phi + \mathrm{i}\omega\,(\boldsymbol{r}\cdot\boldsymbol{p} - \boldsymbol{p}\cdot\boldsymbol{r})\,\phi + \omega^2\boldsymbol{r}^2\phi - 2p_4p_5\phi + k^2\phi = 0 \tag{45}$$

the second term of which is $-3\hbar\omega\phi$, since $xp_x - p_xx = \mathrm{i}\hbar$, etc. Using the embedding, equation (39) and absorbing the constant k into the energy as in equation (40), then we obtain

$$E\phi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega\right)\phi \tag{46}$$

after performing the change $\omega \to m\omega$.

This equation is similar to equation (11) of [23] and has been derived in [1]. The present derivation confirms the idea that the choice of embedding may not be unique.

3.2. *DKP equation for spin one*

As we have mentioned, a DKP equation for spin 1 can be generated by the 15-dimensional representation of $so(5, 1)$: the adjoint representation, easy to construct once the commutation relations, equation (23), are known. Given the commutation relations of some Lie algebra

$$[x_m, x_n] = c^p_{mn}x_p \tag{47}$$

then the entries of the adjoint representation of the element x_m are given by

$$x_m \to [\mathrm{ad}(x_m)]_{pn} \equiv c^p_{mn}. \tag{48}$$

The representation given below is equivalent to the adjoint representation. We do not give explicitly the representation of the whole algebra, but just the elements required, that is, $\beta^\mu = J^{\mu 6}$, with $\mu = 1, \ldots, 5$, also in accordance with equation (21). We use the shorthand notation e_{ij} to represent a 15-by-15 matrix whose only non-zero entry is ij, defined to be one, that is, $(e_{ij})_{mn} \equiv \delta_{im}\delta_{jn}$. Then the DKP generators are

$$\begin{aligned} \beta^1 &= e_{13,1} + e_{14,4} + e_{12,8} - e_{11,9} - e_{9,11} + e_{8,12} + e_{1,13} + e_{4,14} \\ \beta^2 &= e_{13,2} + e_{14,5} - e_{12,7} + e_{10,9} + e_{9,10} - e_{7,12} + e_{2,13} + e_{5,14} \\ \beta^3 &= e_{13,3} + e_{14,6} + e_{11,7} - e_{10,8} - e_{8,10} + e_{7,11} + e_{3,13} + e_{6,14} \\ \beta^4 &= -e_{10,4} - e_{11,5} - e_{12,6} + e_{1,10} + e_{2,11} + e_{3,12} + e_{15,14} + e_{13,15} \\ \beta^5 &= -e_{10,1} - e_{11,2} - e_{12,3} + e_{4,10} + e_{5,11} + e_{6,12} - e_{15,13} - e_{14,15}. \end{aligned} \tag{49}$$

These matrices correspond to the basis elements $J^{\mu 6}$ of $so(5, 1)$ such as given by equation (34). Substituting equation (49) into (25), and with the DKP spinor given by

$$\Psi = \begin{pmatrix} v_1 \\ \vdots \\ v_{15} \end{pmatrix} \tag{50}$$

one obtains the equations

$$\begin{aligned}
&p_4 \boldsymbol{w}_4 - \mathrm{i}k\boldsymbol{w}_1 + \boldsymbol{p}v_{13} = \mathbf{0}\\
&-\mathrm{i}k\boldsymbol{w}_2 + p_5\boldsymbol{w}_4 + \boldsymbol{p}v_{14} = \mathbf{0}\\
&-\mathrm{i}k\boldsymbol{w}_3 - \boldsymbol{p}\times\boldsymbol{w}_4 = \mathbf{0}\\
&-p_4\boldsymbol{w}_2 - p_5\boldsymbol{w}_1 + \boldsymbol{p}\times\boldsymbol{w}_3 - \mathrm{i}k\boldsymbol{w}_4 = \mathbf{0}\\
&p_4v_{15} + \boldsymbol{p}\cdot\boldsymbol{w}_1 - \mathrm{i}kv_{13} = 0\\
&\boldsymbol{p}\cdot\boldsymbol{w}_2 - \mathrm{i}kv_{14} - p_5v_{15} = 0\\
&p_4v_{14} - p_5v_{13} - \mathrm{i}kv_{15} = 0
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\boldsymbol{w}_1 &= (v_1, v_2, v_3)\\
\boldsymbol{w}_2 &= (v_4, v_5, v_6)\\
\boldsymbol{w}_3 &= (v_7, v_8, v_9)\\
\boldsymbol{w}_4 &= (v_{10}, v_{11}, v_{12})\,.
\end{aligned} \tag{52}$$

Let us remark that no embedding has been performed in equation (51). Hereafter we show how it can be reduced to the Schrödinger equation in two different ways: the first method consists in defining the embedding for the fields Ψ while leaving p_4 and p_5 free, whereas the second method consists in solving directly equation (51) but defining p_4 and p_5 at an earlier stage.

Therefore, let us first consider equation (51) with the definitions

$$\begin{aligned}
&\boldsymbol{w}_1 = \boldsymbol{E}\\
&\boldsymbol{w}_2 = -\mathrm{i}p_5\boldsymbol{A}\\
&\boldsymbol{w}_3 = \boldsymbol{B}\\
&\boldsymbol{w}_4 = k\boldsymbol{A}\\
&v_{13} = k\phi \qquad v_{14} = 0 \qquad v_{15} = \mathrm{i}p_5\phi
\end{aligned} \tag{53}$$

or

$$\Psi = \begin{pmatrix} \boldsymbol{E}\\ -\mathrm{i}p_5\boldsymbol{A}\\ \boldsymbol{B}\\ k\boldsymbol{A}\\ k\phi\\ 0\\ \mathrm{i}p_5\phi \end{pmatrix} \tag{54}$$

which suggests an interpretation of the components of the field Ψ in terms of electromagnetic fields. Substituting equation (53) into (51),

$$\begin{aligned}
&p_4\boldsymbol{B} - \mathrm{i}\boldsymbol{E} + \boldsymbol{p}\phi = \mathbf{0}\\
&\mathrm{i}\boldsymbol{B} + \boldsymbol{p}\times\boldsymbol{A} = \mathbf{0}\\
&\mathrm{i}p_4p_5\boldsymbol{A} - p_5\boldsymbol{E} + \boldsymbol{p}\times\boldsymbol{B} - \mathrm{i}k^2\boldsymbol{A} = \mathbf{0}\\
&\mathrm{i}p_4p_5\phi + \boldsymbol{p}\cdot\boldsymbol{E} - \mathrm{i}k^2\phi = 0\\
&\boldsymbol{p}\cdot\boldsymbol{A} + p_5\phi = 0.
\end{aligned} \tag{55}$$

This can be reduced to

$$2p_4p_5\boldsymbol{A} - \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) + \boldsymbol{p}\times\boldsymbol{p}\times\boldsymbol{A} - k^2\boldsymbol{A} = \mathbf{0} \tag{56}$$

and, expressing the penultimate term as $\boldsymbol{p}\times\boldsymbol{p}\times\boldsymbol{A} = \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) - \boldsymbol{p}^2\boldsymbol{A}$ we finally obtain

$$E\boldsymbol{A} = \frac{\boldsymbol{p}^2}{2m}\boldsymbol{A} \tag{57}$$

where we have used equation (39).

The second way to obtain equation (57) is first by adding together the first two lines of equation (51) so that we find

$$\mathrm{i}k(\boldsymbol{w}_1 + \boldsymbol{w}_2) = (p_4 + p_5)\boldsymbol{w}_4 + \boldsymbol{p}(v_{13} + v_{14}) \tag{58}$$

and if we add together the fifth and sixth lines then we find

$$\boldsymbol{p} \cdot (\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathrm{i}k(v_{13} + v_{14}) - (p_4 - p_5)v_{15}. \tag{59}$$

If we put $p_4 = p_5$ and $v_{13} + v_{14} = 0$ then the two previous equations lead to

$$\boldsymbol{p} \cdot \boldsymbol{w}_4 = 0. \tag{60}$$

Next we substitute the third line of equation (51) into the fourth line to obtain

$$-\mathrm{i}k(p_4\boldsymbol{w}_2 + p_5\boldsymbol{w}_1) - \boldsymbol{p} \times \boldsymbol{p} \times \boldsymbol{w}_4 + k^2\boldsymbol{w}_4 = \boldsymbol{0} \tag{61}$$

and using equation (58), with our choices for p_4, p_5 and $v_{13} + v_{14}$, we find

$$-2p_4p_5\boldsymbol{w}_4 - \boldsymbol{p} \times \boldsymbol{p} \times \boldsymbol{w}_4 + k^2\boldsymbol{w}_4 = \boldsymbol{0}. \tag{62}$$

If we select the embedding for p_4 and p_5 as in equation (39), define $\boldsymbol{A} \equiv \boldsymbol{w}_4$, use equation (60) and include the k^2 into the energy as in (40), then (62) is reduced to (57).

Next we consider the DKP simple harmonic oscillator by first performing the non-minimal substitution, equation (42), where η is given by equations (29) and (49):

$$\begin{aligned}
&p_4\boldsymbol{w}_4 - \mathrm{i}k\boldsymbol{w}_1 + \boldsymbol{p}v_{13} + \mathrm{i}\omega v_{14}\boldsymbol{r} = \boldsymbol{0}\\
&-\mathrm{i}k\boldsymbol{w}_2 + p_5\boldsymbol{w}_4 + \boldsymbol{p}v_{14} + \mathrm{i}\omega v_{13}\boldsymbol{r} = \boldsymbol{0}\\
&-\mathrm{i}k\boldsymbol{w}_3 - \boldsymbol{p}_- \times \boldsymbol{w}_4 = \boldsymbol{0}\\
&-p_4\boldsymbol{w}_2 - p_5\boldsymbol{w}_1 + \boldsymbol{p}_+ \times \boldsymbol{w}_3 - \mathrm{i}k\boldsymbol{w}_4 = \boldsymbol{0}\\
&p_4v_{15} + \boldsymbol{p} \cdot \boldsymbol{w}_1 - \mathrm{i}kv_{13} - \mathrm{i}\omega\boldsymbol{r} \cdot \boldsymbol{w}_2 = 0\\
&\boldsymbol{p} \cdot \boldsymbol{w}_2 - \mathrm{i}kv_{14} - p_5v_{15} - \mathrm{i}\omega\boldsymbol{r} \cdot \boldsymbol{w}_1 = 0\\
&p_4v_{14} - p_5v_{13} - \mathrm{i}kv_{15} = 0
\end{aligned} \tag{63}$$

where we have used the shorthand notation

$$\boldsymbol{p}_\pm \equiv \boldsymbol{p} \pm \mathrm{i}\omega\boldsymbol{r}. \tag{64}$$

Let us recall that a factor m is going to be included within ω at the end of the calculations.

Here again we shall display two ways to obtain equation (67). First we substitute the embedding defined in equation (53) into (63) to get

$$\begin{aligned}
&p_4\boldsymbol{A} - \mathrm{i}\boldsymbol{E} + \boldsymbol{p}\phi = \boldsymbol{0}\\
&\mathrm{i}\omega\boldsymbol{r}\phi = \boldsymbol{0}\\
&\mathrm{i}\boldsymbol{B} + \boldsymbol{p}_- \times \boldsymbol{A} = \boldsymbol{0}\\
&\mathrm{i}p_4p_5\boldsymbol{A} - p_5\boldsymbol{E} + \boldsymbol{p}_+ \times \boldsymbol{B} - \mathrm{i}k^2\boldsymbol{A} = \boldsymbol{0}\\
&\mathrm{i}p_4p_5\phi + \boldsymbol{p} \cdot \boldsymbol{E} - \mathrm{i}k^2\phi - p_5\omega\boldsymbol{r} \cdot \boldsymbol{A} = 0\\
&p_5\boldsymbol{p} \cdot \boldsymbol{A} + p_5^2\phi + \omega\boldsymbol{r} \cdot \boldsymbol{E} = 0.
\end{aligned} \tag{65}$$

From the second line, $\phi = 0$, and substituting the first and third lines into the fourth line of equation (65), we find

$$2p_4p_5\boldsymbol{A} + \boldsymbol{p}_+ \times \boldsymbol{p}_- \times \boldsymbol{A} - k^2\boldsymbol{A} = \boldsymbol{0} \tag{66}$$

from which, by using equations (39) and (40), we have

$$E\boldsymbol{A} = -\frac{1}{2m}\boldsymbol{p}_+ \times (\boldsymbol{p}_- \times \boldsymbol{A}). \tag{67}$$

Before developing this expression further, let us show how it can be obtained also without defining the embedding of the fields as in equation (53). From the third and fourth lines of equation (63) we get

$$\mathrm{i}kp_5\boldsymbol{w}_1 + \mathrm{i}kp_4\boldsymbol{w}_2 - k^2\boldsymbol{w}_4 + \boldsymbol{p}_+ \times (\boldsymbol{p}_- \times \boldsymbol{w}_4) = \boldsymbol{0}. \tag{68}$$

Next we add together the first and second lines of equation (63) to obtain

$$\mathrm{i}k(\boldsymbol{w}_1 + \boldsymbol{w}_2) = (p_4 + p_5)\boldsymbol{w}_4 + \boldsymbol{p}_+(v_{13} + v_{14}) \tag{69}$$

and adding the fifth and sixth lines gives

$$\boldsymbol{p}_- \cdot (\boldsymbol{w}_1 + \boldsymbol{w}_2) - \mathrm{i}k(v_{13} + v_{14}) + (p_4 - p_5)v_{15} = 0. \tag{70}$$

As for the free case, this suggests to define the embedding such that $p_4 = p_5$ and $v_{13}+v_{14} = 0$. Then from equations (69) and (70) one obtains the orthogonality condition

$$\boldsymbol{p}_- \cdot \boldsymbol{w}_4 = 0 \rightarrow \boldsymbol{p}_- \cdot \boldsymbol{A} = 0 \tag{71}$$

if we define $\boldsymbol{A} \equiv \boldsymbol{w}_4$ as done previously. Then equation (68) becomes

$$p_4(2p_5)\boldsymbol{w}_4 - k^2\boldsymbol{w}_4 + \boldsymbol{p}_+ \times (\boldsymbol{p}_- \times \boldsymbol{w}_4) = \boldsymbol{0} \tag{72}$$

from which we find

$$(2p_4p_5 - k^2)\boldsymbol{w}_4 = -\boldsymbol{p}_+ \times (\boldsymbol{p}_- \times \boldsymbol{w}_4). \tag{73}$$

Using once again the embedding in equation (39) and the redefinition in equation (40) we obtain equation (67), as expected.

It is shown in appendix A that equation (67) leads to the Schrödinger equation for a harmonic oscillator including the effect of spin–orbit coupling:

$$E\boldsymbol{A} = \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{\omega}{\hbar}\boldsymbol{L}\cdot\boldsymbol{S}\right]\boldsymbol{A}. \tag{74}$$

This is the non-relativistic version obtained earlier (equation (16) in [23]). It should be emphasized that both spin 0 and spin 1 require the same embedding, equation (39). We do not have to specify p_4 and p_5 separately. In the next section, we will see that the embedding has to be modified for spin one-half.

It is worth noting that with the non-minimal coupling, equation (64), the Schrödinger equation for the harmonic oscillator contains the spin–orbit interaction. As noted by Nikitin and Fuschich [20], the dipole, spin–orbit, quadrupole and Darwin couplings of the particle to an external electromagnetic field can be derived with a proper minimal coupling $p_\mu \rightarrow p_\mu - \mathrm{i}eA_\mu$, where A_μ is the vector potential for the external field. It is important to have the vector potential transform as $A_i' = R_i^j A_j$ and $A_0' = A_0 + v_j A_j$, where R_j^i is a rotation matrix and v_j is the velocity of the particle. It is important to emphasize that the additional terms are usually considered to be a direct consequence of taking the non-relativistic limit to order $1/m^2$ of the Lorentz-invariant equation. Here we have corroborated the results of Nikitin and Fuschich partly, by the appearance of spin–orbit interaction, within a Galilei-covariant approach that makes the formulation similar to the Lorentz-covariant approach.

4. Dirac equation: spin 1/2

As stated in the introduction, the algorithm described in the previous sections provides us with a representation of the non-relativistic Dirac equation. Such a Dirac-like equation has been constructed in [4]. Here our purpose is to treat it in the same context as the equations for integer spin and illustrate how the formalism allows us to recover the non-relativistic limit of the Dirac oscillator [24].

The non-relativistic Dirac equation is

$$\left(\gamma^{\mu}\partial_{\mu}+k\right)\Psi=0 \qquad \mu=1,\ldots,5 \tag{75}$$

written in momentum space as equation (25) with the α replaced by the γ, which satisfy

$$\{\gamma^{\mu},\gamma^{\nu}\}=\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}. \tag{76}$$

The gamma matrices can be chosen as

$$\gamma^{n}=\begin{pmatrix}\sigma_n & 0\\ 0 & -\sigma_n\end{pmatrix} \qquad \gamma^{4}=\begin{pmatrix}0 & 0\\ -\sqrt{2} & 0\end{pmatrix} \qquad \gamma^{5}=\begin{pmatrix}0 & \sqrt{2}\\ 0 & 0\end{pmatrix} \tag{77}$$

where each entry is a two-by-two matrix and the σ_n are the spin Pauli matrices. As for the integer spins, they generate the Lie algebra $so(5,1)$ by taking, as in equation (21),

$$\gamma^{\mu}=J^{\mu 6} \qquad \mu=1,\ldots,5 \tag{78}$$

and then using equation (23) to generate the remaining J. The adjoint spinor is defined as $\overline{\Psi}=\Psi^{\dagger}\zeta$, with

$$\zeta=\frac{-\mathrm{i}}{\sqrt{2}}\left(\gamma^{4}+\gamma^{5}\right)=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}. \tag{79}$$

The matrix ζ plays a role similar to η, equation (29), in DKP equations and will be used hereafter to investigate the harmonic oscillator. Note that when we act on the left of equation (75) with $(\gamma^{\mu}\partial_{\mu}-k)$,

$$\begin{aligned}0&=\left(\gamma^{\mu}\partial_{\mu}-k\right)\left(\gamma^{\nu}\partial_{\nu}+k\right)\Psi\\&=\left(\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}-k^{2}\right)\Psi\\&=\left(\tfrac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\}\partial_{\mu}\partial_{\nu}-k^{2}\right)\Psi\\&=\left(g^{\mu\nu}\partial_{\mu}\partial_{\nu}-k^{2}\right)\Psi\end{aligned} \tag{80}$$

which can be seen as a non-relativistic Klein–Gordon equation to be satisfied by each field component, as for the relativistic field counterpart. This provides a constraint on the spin half Galilei covariant equation as was the case for the equations obtained by Dirac, Fierz and Pauli. The mathematical structure is quite similar. However in the present case k is not the mass. This equation has direct relevance to the presence of phonons in crystals.

Now let us consider the harmonic oscillator. If we write the Bhabha equation using the representation, equation (77), then performing the non-minimal substitution analogous to equation (42), with η now replaced by ζ, for a field

$$\Psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{81}$$

one finds

$$\begin{aligned}&(\boldsymbol{\sigma}\cdot\boldsymbol{p}-\mathrm{i}k)\,\varphi+\left(\omega\boldsymbol{\sigma}\cdot\boldsymbol{r}+\sqrt{2}p_5\right)\chi=0\\&(\boldsymbol{\sigma}\cdot\boldsymbol{p}+\mathrm{i}k)\,\chi+\left(\sqrt{2}p_4-\omega\boldsymbol{\sigma}\cdot\boldsymbol{r}\right)\varphi=0.\end{aligned} \tag{82}$$

In appendix B, we have gathered the calculations leading from equation (82) to the Schrödinger equation for a harmonic oscillator including the spin–orbit coupling

$$E\varphi=\left(\frac{\boldsymbol{p}^{2}}{2m}+\frac{1}{2}m\omega^{2}\boldsymbol{r}^{2}-\frac{3}{2}\hbar\omega-\frac{2}{\hbar}\omega\boldsymbol{L}\cdot\boldsymbol{S}\right)\varphi. \tag{83}$$

Obviously a similar equation can be obtained in terms of the field χ. This result, equation (83), is in agreement with the non-relativistic limit of the Dirac oscillator investigated in [24].

5. Concluding remarks

In this paper, we have applied a Galilean covariant formalism in five dimensions to construct non-relativistic first-order Bhabha wave equations. We have considered particles with spin 0, 1/2 and 1, having obtained in each case the Hamiltonian of the harmonic oscillator. Higher spins (3/2, 2, 5/2, ...) can be studied using this approach and are included in other irreducible representations of $so(5, 1)$.

Related topics can be further investigated: (1) the two non-relativistic limits of the electromagnetic field [25] in a covariant way; (2) description of the spin content of various irreducible representations of $so(5, 1)$ as done in the paper II of the series [10] for the relativistic case; (3) introduction of gauge fields into wave equations for various spins; (4) investigation of the wave equation for spin 2 particles and its applications to gravitational fields; and (5) quantization of the non-relativistic Galilei-covariant Bhabha equations. Other projects involving Galilean covariance but not specifically related to linear wave equations are the investigation of equations for fluids and for the kinetic theory (like the Vlasov and Boltzmann equations), the study of Galilean supersymmetry, and applications in thermo field dynamics.

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Appendix A. Spin one

This appendix contains the details of the calculations starting with equation (67) and leading to equation (74). The first step involves a careful manipulation of the triple product in equation (67); indeed one must keep in mind that $\boldsymbol{p}_\pm$ involves operators that act on the field $\boldsymbol{A}$. Therefore one must use an identity such that the vector field $\boldsymbol{A}$ always lies at the far right:

$$(\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}))_m = b_m\,(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\,c_m + \sum_{n=1}^{3} [a_n, b_m]\,c_n. \tag{A.1}$$

From this equation, the triple product of equation (67) becomes

$$\begin{aligned} \boldsymbol{p}_+ \times (\boldsymbol{p}_- \times \boldsymbol{A}) &= \boldsymbol{p}_-(\boldsymbol{p}_+ \cdot \boldsymbol{A}) - (\boldsymbol{p}_+ \cdot \boldsymbol{p}_-)\boldsymbol{A} + \sum_{n=1}^{3}[(\boldsymbol{p}_+)_n, (\boldsymbol{p}_-)_m]A_n \\ &= \boldsymbol{p}_-(\boldsymbol{p}_+ \cdot \boldsymbol{A}) - (\boldsymbol{p}_+ \cdot \boldsymbol{p}_-)\boldsymbol{A} - 2\hbar\omega\boldsymbol{A} \end{aligned} \tag{A.2}$$

where we have used $[(\boldsymbol{p}_+)_m, (\boldsymbol{p}_-)_n] = -2\hbar\omega\delta_{mn}$.

Next let us expand the first term of equation (A.2):

$$\begin{aligned} \boldsymbol{p}_-(\boldsymbol{p}_+ \cdot \boldsymbol{A}) &= (\boldsymbol{p} - \mathrm{i}\omega\boldsymbol{r})\,[(\boldsymbol{p} + \mathrm{i}\omega\boldsymbol{r}) \cdot \boldsymbol{A}] \\ &= \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{A}) + \omega^2\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{A}) - \mathrm{i}\omega\,[\boldsymbol{r}(\boldsymbol{p} \cdot \boldsymbol{A}) - \boldsymbol{p}(\boldsymbol{r} \cdot \boldsymbol{A})] \\ &= \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{A}) + \omega^2\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{A}) + \omega[\hbar + \boldsymbol{L} \cdot \boldsymbol{S}]\boldsymbol{A}. \end{aligned} \tag{A.3}$$

The last line has been obtained by first developing the last term of the second line of equation (A.3) as

$$\begin{aligned} [\boldsymbol{r}(\boldsymbol{p} \cdot \boldsymbol{A}) - \boldsymbol{p}(\boldsymbol{r} \cdot \boldsymbol{A})]_m &= \sum_n r_m p_n A_n - p_m r_n A_n \\ &= -\sum_n (r_n p_m - r_m p_n) A_n + \mathrm{i}\hbar A_m \\ &= -(\boldsymbol{L} \times \boldsymbol{A})_m + \mathrm{i}\hbar A_m. \end{aligned} \tag{A.4}$$

(In the second line we have used the canonical commutation relations: $r_m p_n - p_n r_m = \mathrm{i}\hbar\delta_{mn}$.) Next we consider the spin 1 representation for which $(S_m)_{kl} = -\mathrm{i}\hbar\epsilon_{klm}$ (as in equation (A.1) of [23]) and then calculate

$$\begin{aligned}
\mathrm{i}(\boldsymbol{L}\times\boldsymbol{A})_m &= \mathrm{i}\epsilon_{mnp}L_n A_p \\
&= -\mathrm{i}\epsilon_{mpn}L_n A_p \\
&= \frac{1}{\hbar}L_n(S_n)_{mp}A_p \\
&= \frac{1}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})_{mp}A_p \\
&= \frac{1}{\hbar}\,[(\boldsymbol{L}\cdot\boldsymbol{S})\boldsymbol{A}]_m
\end{aligned} \tag{A.5}$$

so that $\boldsymbol{L}\times\boldsymbol{A} = -\frac{\mathrm{i}}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})\boldsymbol{A}$.

Having obtained equation (A.3) now we expand the second term of equation (A.2):

$$\begin{aligned}
(\boldsymbol{p}_+\cdot\boldsymbol{p}_-)\boldsymbol{A} &= (\boldsymbol{p}+\mathrm{i}\omega\boldsymbol{r})\cdot(\boldsymbol{p}-\mathrm{i}\omega\boldsymbol{r})\boldsymbol{A} \\
&= [\boldsymbol{p}^2 + \mathrm{i}\omega(\boldsymbol{r}\cdot\boldsymbol{p} - \boldsymbol{p}\cdot\boldsymbol{r}) + \omega^2\boldsymbol{r}^2]\boldsymbol{A} \\
&= [\boldsymbol{p}^2 + \mathrm{i}\omega(3\mathrm{i}\hbar) + \omega^2\boldsymbol{r}^2]\boldsymbol{A} \\
&= \boldsymbol{p}^2\boldsymbol{A} - 3\hbar\omega\boldsymbol{A} + \omega^2\boldsymbol{r}^2\boldsymbol{A}.
\end{aligned} \tag{A.6}$$

Substituting equations (A.3) and (A.6) back into (A.2) we find

$$\begin{aligned}
\boldsymbol{p}_+\times(\boldsymbol{p}_-\times\boldsymbol{A}) &= \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) + \omega^2\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) + \hbar\omega\boldsymbol{A} + \frac{\omega}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})\boldsymbol{A} \\
&\quad -\boldsymbol{p}^2\boldsymbol{A} + 3\hbar\omega\boldsymbol{A} - \omega^2\boldsymbol{r}^2\boldsymbol{A} - 2\hbar\omega\boldsymbol{A} \\
&= \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) - \boldsymbol{p}^2\boldsymbol{A} + \omega^2[\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) - \boldsymbol{r}^2\boldsymbol{A}] + \omega\left(2\hbar + \frac{1}{\hbar}\boldsymbol{L}\cdot\boldsymbol{S}\right)\boldsymbol{A}
\end{aligned} \tag{A.7}$$

which corresponds to equation (A.2) of [23].

The last step consists in using equation (71), which is reminiscent of some sort of gauge fixing. From this one can assert that

$$\begin{aligned}
0 &= \boldsymbol{p}_+(\boldsymbol{p}_-\cdot\boldsymbol{A}) \\
&= \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) - \mathrm{i}\omega\boldsymbol{p}(\boldsymbol{r}\cdot\boldsymbol{A}) + \mathrm{i}\omega\boldsymbol{r}(\boldsymbol{p}\cdot\boldsymbol{A}) + \omega^2\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) \\
&= \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) - \frac{\omega}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})\boldsymbol{A} - \hbar\omega\boldsymbol{A} + \omega^2\boldsymbol{r}\cdot\boldsymbol{A}
\end{aligned} \tag{A.8}$$

where we have used once again equations (A.4) and (A.5). Using equations (A.8), (A.7) and redefining $\omega \to m\omega$, we can rewrite equation (67) as

$$\begin{aligned}
E\boldsymbol{A} &= -\frac{1}{2m}\boldsymbol{p}_+\times(\boldsymbol{p}_-\times\boldsymbol{A}) \\
&= \frac{1}{2m}\left[\boldsymbol{p}_+(\boldsymbol{p}_-\cdot\boldsymbol{A}) - \boldsymbol{p}_+\times(\boldsymbol{p}_-\times\boldsymbol{A})\right] \\
&= \frac{1}{2m}\Bigg[\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) + m^2\omega^2\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) - \frac{m}{\hbar}\omega(\boldsymbol{L}\cdot\boldsymbol{S})\boldsymbol{A} - m\hbar\omega\boldsymbol{A} - \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) \\
&\quad +\boldsymbol{p}^2\boldsymbol{A} - m^2\omega^2\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) + m^2\omega^2\boldsymbol{r}^2\boldsymbol{A} - 2m\hbar\omega\boldsymbol{A} - \frac{1}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})m\omega\boldsymbol{A}\Bigg] \\
&= \frac{1}{2m}\left[\boldsymbol{p}^2 - 2\frac{m}{\hbar}\omega\boldsymbol{L}\cdot\boldsymbol{S} - 3m\hbar\omega + m^2\omega^2\boldsymbol{r}^2\right]\boldsymbol{A} \\
&= \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{\omega}{\hbar}\boldsymbol{L}\cdot\boldsymbol{S}\right]\boldsymbol{A}.
\end{aligned} \tag{A.9}$$

Appendix B. Spin half

In this section we show how the equation (83) can be obtained from equation (82). Multiplying the first line of equation (82) by $\boldsymbol{\sigma}\cdot\boldsymbol{p}+\mathrm{i}k$ on the left, and the second line by $-\left(\sqrt{2}p_5+\omega\boldsymbol{\sigma}\cdot\boldsymbol{r}\right)$, and adding the two resulting equations together gives

$$\left[\boldsymbol{p}^2+k^2+\omega^2\boldsymbol{r}^2-2p_4p_5-\sqrt{2}\omega\boldsymbol{\sigma}\cdot\boldsymbol{r}(p_4-p_5)\right]\varphi \\ -\omega[(\boldsymbol{\sigma}\cdot\boldsymbol{r})(\boldsymbol{\sigma}\cdot\boldsymbol{p})-(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{r})]\chi=0. \tag{B.1}$$

The identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \tag{B.2}$$

has been used repeatedly. As stated before, we see from equation (B.1) that here we are forced to choose an embedding such that $p_4=p_5$, unlike the integer spins. Now we use

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{r})(\boldsymbol{\sigma}\cdot\boldsymbol{p})-(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{r})&=\boldsymbol{r}\cdot\boldsymbol{p}+\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{r}\times\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{r}+\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{r}\\&=(\boldsymbol{r}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{r})+2\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{L}\\&=3\mathrm{i}\hbar+2\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{L}\end{aligned} \tag{B.3}$$

together with the embedding provided by equation (39) and

$$\chi=-\mathrm{i}\varphi \tag{B.4}$$

to obtain

$$2mE\varphi=\left(\boldsymbol{p}^2+k^2+\omega^2\boldsymbol{r}^2-3\hbar\omega-2\omega\boldsymbol{\sigma}\cdot\boldsymbol{L}\right)\varphi. \tag{B.5}$$

By redefining the energy as in equation (40) and including explicitly the mass m into ω, we find

$$E\varphi=\left(\frac{\boldsymbol{p}^2}{2m}+\frac{1}{2}m\omega^2\boldsymbol{r}^2-\frac{3}{2}\hbar\omega-\frac{2}{\hbar}\omega\boldsymbol{L}\cdot\boldsymbol{S}\right)\varphi \tag{B.6}$$

where we have used the usual definition $\boldsymbol{S}\equiv\frac{1}{2}\hbar\boldsymbol{\sigma}$.

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